Tractable Profit Maximization over Multiple Attributes under Discrete Choice Models

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Agenda

What is “profit maximization over *multiple attributes*”?

Why should we care about it?

How can we solve it efficiently?

What is “profit maximization over multiple attributes”?

Pricing Problems: A Minimal Example

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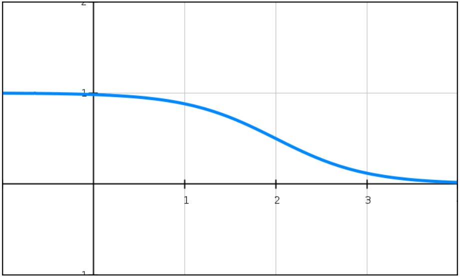
Consider selling a product to a group of customers:

- Purchasing probability is a (decreasing) function of price;



Example:
choice model

$$P(x) = \frac{e^{4-2x}}{1 + e^{4-2x}}$$

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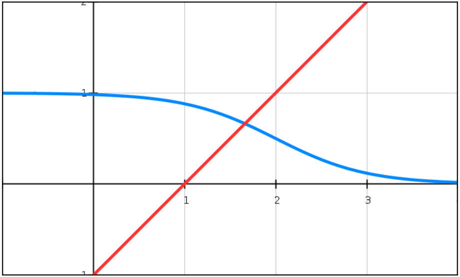

Example:
cost = 1

$$f(x) = x - 1$$

Pricing Problems: A Minimal Example

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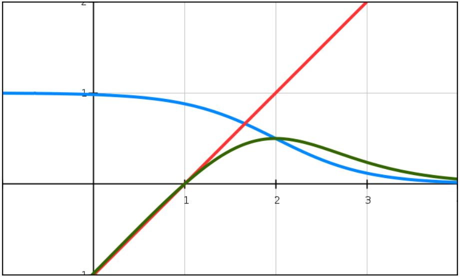
- Profit margin is a (increasing) (linear) function of price;
- Expected profit is the product of the two terms.



Expected profit:
(per customer visit)

$$f(x)P(x)$$

Generalizations to the Classical Pricing Problem

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Example: cancellation fee of airline tickets

- High fees lead to low choice probabilities (profit margin $\neq 1$);
- However, a higher cancellation fee reduces operational costs;

The operator is now facing a profit maximization problem where both price and cancellation fee are decision variables.

To make it concrete:

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Static Attribute Optimization:

$$
\begin{aligned}
\max_{y} \quad & \sum_{j \in \mathcal{J}} \left(\sum_{k \in \mathcal{K}_j} \check{\phi}_k y_{jk} - \check{\psi}_j \right) \check{P}_j(y) & \\
\text{s.t.} \quad & y_{jk} \geq \underline{y}_{jk} & \forall\, j \in \mathcal{J}, k \in \underline{\mathcal{K}}_j \\
& y_{jk} \leq \overline{y}_{jk} & \forall\, j \in \mathcal{J}, k \in \overline{\mathcal{K}}_j
\end{aligned}
$$

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- Profit margin of product j is $\sum_{k \in \mathcal{K}_j} \check{\phi}_k y_{jk} - \check{\psi}_j$.

Revenue Management (Fluid Optimization):

$$\begin{aligned}
\max_{y} \quad & \sum_{t=0}^{T} \lambda_t \sum_{j \in \mathcal{J}} \left(\sum_{k \in \mathcal{K}_j} \check{\phi}_{kt} y_{jkt} - \check{\psi}_{jt} \right) \check{P}_{jt}(y) \\
\text{s.t.} \quad & \sum_{t=0}^{T} \lambda_t \sum_{j \in \mathcal{J}} a_{rj} \check{P}_{jt}(y) \quad \leq \quad b_r && \forall\ r \in \mathcal{R} \\
& y_{jkt} \quad \geq \quad \underline{y}_{jkt} && \forall\ j \in \mathcal{J}, k \in \underline{\mathcal{K}}_j, t = 0, \ldots, T \\
& y_{jkt} \quad \leq \quad \overline{y}_{jkt} && \forall\ j \in \mathcal{J}, k \in \overline{\mathcal{K}}_j, t = 1, \ldots, T
\end{aligned}$$

Why should we care about it?

Application: Examples

Such problems commonly exist:

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- hotel chain: (1) price, (2) cancelation policy, (3) check-in time, and (4) check-out time.

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- hotel chain: (1) price, (2) cancelation policy, (3) check-in time, and (4) check-out time.

Special cases of this problem exist in literature, but a general discussion is still needed.

Tractability: Approaches

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Method 1: Market Share Approach

- [Song and Xue, 2007] [Dong et al., 2009] [Li and Huh, 2011];
- Prices as inverse functions of market shares: $y_j = P^{-1}(d)$;
- Expected profit is concave in market shares (MNL, NL, ...);

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- Prices as inverse functions of market shares: $y_j = P^{-1}(d)$;
- Expected profit is concave in market shares (MNL, NL, ...);
- Inverse function doesn't exist with multiple attributes.

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Method 2: Constant Adjusted Markup

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Method 2: Constant Adjusted Markup

- [Wang, 2012] [Gallego and Wang, 2014];
- Use optimality conditions: "Constant adjusted markup";
- Reduce the problem to single-dimensional; Unimodularity;

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Method 2: Constant Adjusted Markup

- [Wang, 2012] [Gallego and Wang, 2014];
- Use optimality conditions: "Constant adjusted markup";
- Reduce the problem to single-dimensional; Unimodularity;
- Does not work for constrained problems.

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Method 3: Assortment Planning

- [Davis et al., 2013]: totally uni-modular constraints;
- Discretize price and generate candidate "products";
- One candidate for each product, forced by constraints;

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- Discretize price and generate candidate "products";
- One candidate for each product, forced by constraints;
- With multiple attributes, the number of candidate attribute vectors can be very large.

A tractable solution method still needs to be developed.

How can we solve it efficiently?

Example: MNL

We developed a tractable method (conic transformation) to solve the problem under the MNL, MC and NL models.

Example: MNL

Recall:

$$\max_{y} \quad \sum_{j \in \mathcal{J}} \left(\sum_{k \in \mathcal{K}_j} \check{\phi}_k y_{jk} - \check{\psi}_j \right) \check{P}_j(y) \tag{SP}$$

$$\text{s.t.} \quad y_{jk} \geq \underline{y}_{jk} \qquad \forall\, j \in \mathcal{J}, k \in \underline{\mathcal{K}}_j$$

$$y_{jk} \leq \overline{y}_{jk} \qquad \forall\, j \in \mathcal{J}, k \in \overline{\mathcal{K}}_j$$

Example: MNL

Recall:

$$\begin{aligned} \max_{y} \quad & \sum_{j \in \mathcal{J}} \left(\sum_{k \in \mathcal{K}_j} \check{\phi}_k y_{jk} - \check{\psi}_j \right) \check{P}_j(y) && \text{(SP)} \\ \text{s.t.} \quad & y_{jk} \geq \underline{y}_{jk} && \forall\, j \in \mathcal{J}, k \in \underline{\mathcal{K}}_j \\ & y_{jk} \leq \overline{y}_{jk} && \forall\, j \in \mathcal{J}, k \in \overline{\mathcal{K}}_j \end{aligned}$$

Let $\check{P}_j(y)$ be given by a multinomial logit model:

$$\check{P}_j(y) = \frac{\exp\left(\alpha_j - \sum_{k \in \mathcal{K}_j} \beta_k y_{jk}\right)}{1 + \sum_{j' \in \mathcal{J}} \exp\left(\alpha_{j'} - \sum_{k \in \mathcal{K}_{j'}} \beta_k y_{j'k}\right)}$$

Example: MNL

Recall:

$$\max_{y} \quad \sum_{j \in \mathcal{J}} \left(\sum_{k \in \mathcal{K}_j} \check{\phi}_k y_{jk} - \check{\psi}_j \right) \check{P}_j(y) \qquad \text{(SP)}$$

$$\text{s.t.} \quad y_{jk} \geq \underline{y}_{jk} \qquad \forall\, j \in \mathcal{J}, k \in \underline{\mathcal{K}}_j$$

$$y_{jk} \leq \overline{y}_{jk} \qquad \forall\, j \in \mathcal{J}, k \in \overline{\mathcal{K}}_j$$

Consider variable transformation $x_{jk} = \beta_k y_{jk} - \alpha_j / K_j$**:**

$$P_j(x) = \frac{\exp\left(-\sum_{k \in \mathcal{K}_j} x_{jk}\right)}{1 + \sum_{j' \in \mathcal{J}} \exp\left(\sum_{k \in \mathcal{K}_{j'}} x_{j'k}\right)}$$

Example: MNL

Recall:

$$\max_{d,\, x} \quad \sum_{j \in \mathcal{J}} \left(\sum_{k \in \mathcal{K}_j} \phi_k x_{jk} - \psi_j \right) P_j(x) \qquad (\mathrm{SP}_1^{\mathrm{MNL}})$$

$$\text{s.t.} \quad x_{jk} \geq \underline{x}_{jk} \qquad \forall\, j \in \mathcal{J}, k \in \underline{\mathcal{K}}_j \qquad (1)$$

$$x_{jk} \leq \overline{x}_{jk} \qquad \forall\, j \in \mathcal{J}, k \in \overline{\mathcal{K}}_j \qquad (2)$$

Consider variable transformation $x_{jk} = \beta_k y_{jk} - \alpha_j / K_j$:

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Example: MNL

(SP) becomes:

$$\max_{d,\,x} \quad \sum_{j\in\mathcal{J}} \left(\sum_{k\in\mathcal{K}_j} \phi_k x_{jk} - \psi_j \right) d_j \qquad (\text{SP}_1^{\text{MNL}})$$

$$\text{s.t.} \quad d_j = \frac{\exp\left(-\sum_{k\in\mathcal{K}_j} x_{jk}\right)}{1+\sum_{j'\in\mathcal{J}} \exp\left(-\sum_{k\in\mathcal{K}_{j'}} x_{j'k}\right)} \qquad \forall\, j\in\mathcal{J} \qquad (1)$$

$$x_{jk} \geq \underline{x}_{jk} \qquad \forall\, j\in\mathcal{J}, k\in\underline{\mathcal{K}}_j \qquad (2)$$

$$x_{jk} \leq \overline{x}_{jk} \qquad \forall\, j\in\mathcal{J}, k\in\overline{\mathcal{K}}_j \qquad (3)$$

Example: MNL

We consider ..

$$\max_{d_0, d, x} \quad \sum_{j \in \mathcal{J}} \left(\sum_{k \in \mathcal{K}_j} \phi_k x_{jk} - \psi_j \right) d_j \qquad (\mathrm{SP}_1^{\mathrm{MNL}})$$

$$\text{s.t.} \quad d_j = \frac{\exp\left(-\sum_{k \in \mathcal{K}_j} x_{jk}\right)}{1 + \sum_{j' \in \mathcal{J}} \exp\left(-\sum_{k \in \mathcal{K}_{j'}} x_{j'k}\right)} \qquad \forall\, j \in \mathcal{J} \qquad (1)$$

$$d_0 = \frac{1}{1 + \sum_{j' \in \mathcal{J}} \exp\left(-\sum_{k \in \mathcal{K}_{j'}} x_{j'k}\right)} \qquad (\text{no-purchase prob.})$$

$$x_{jk} \geq \underline{x}_{jk} \qquad \forall\, j \in \mathcal{J}, k \in \underline{\mathcal{K}}_j \qquad (2)$$

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Example: MNL

We consider .. with dummy variable d_0

$$\max_{d_0, d, x} \quad \sum_{j \in \mathcal{J}} \left(\sum_{k \in \mathcal{K}_j} \phi_k x_{jk} - \psi_j \right) d_j \qquad (\text{SP}_1^{\text{MNL}})$$

$$\text{s.t.} \quad d_j = \frac{\exp\left(-\sum_{k \in \mathcal{K}_j} x_{jk}\right)}{1 + \sum_{j' \in \mathcal{J}} \exp\left(-\sum_{k \in \mathcal{K}_{j'}} x_{j'k}\right)} \qquad \forall\, j \in \mathcal{J} \qquad (1)$$

$$d_0 = \frac{1}{1 + \sum_{j' \in \mathcal{J}} \exp\left(-\sum_{k \in \mathcal{K}_{j'}} x_{j'k}\right)} \qquad (\text{no-purchase prob.})$$

$$x_{jk} \geq \underline{x}_{jk} \qquad \forall\, j \in \mathcal{J}, k \in \underline{\mathcal{K}}_j \qquad (2)$$

$$x_{jk} \leq \overline{x}_{jk} \qquad \forall\, j \in \mathcal{J}, k \in \overline{\mathcal{K}}_j \qquad (3)$$

Assumptions: (1) $\phi > 0$**; (2) no "unlimited trade-off"**

Convex Transformation

Convex Transformation

Consider:

$$
\begin{aligned}
\max_{d_0, d, x} \quad & \sum_{j \in \mathcal{J}} \left(\sum_{k \in \mathcal{K}_j} \phi_k x_{jk} - \psi_j \right) d_j && (\mathrm{SP}_1^{\mathrm{MNL}}) \\
\text{s.t.} \quad & d_j = \frac{\exp\left(-\sum_{k \in \mathcal{K}_j} x_{jk}\right)}{1 + \sum_{j' \in \mathcal{J}} \exp\left(-\sum_{k \in \mathcal{K}_{j'}} x_{j'k}\right)} && \forall\, j \in \mathcal{J} \\
& d_0 = \frac{1}{1 + \sum_{j' \in \mathcal{J}} \exp\left(-\sum_{k \in \mathcal{K}_{j'}} x_{j'k}\right)} \\
& x_{jk} \geq \underline{x}_{jk} && \forall\, j \in \mathcal{J}, k \in \underline{\mathcal{K}}_j \\
& x_{jk} \leq \overline{x}_{jk} && \forall\, j \in \mathcal{J}, k \in \overline{\mathcal{K}}_j
\end{aligned}
$$

Convex Transformation

The problem is equivalent to:

$$\max_{d,\, d_0,\, x} \quad \sum_{j \in \mathcal{J}} \left(\sum_{k \in \mathcal{K}_j} \phi_k x_{jk} - \psi_j \right) d_j \qquad (\mathrm{SP}_2^{\mathrm{MNL}})$$

$$\text{s.t.} \quad \ln\left(\frac{d_j}{d_0}\right) = -\sum_{k \in \mathcal{K}_j} x_{jk} \qquad \forall\, j \in \mathcal{J}$$

$$x_{jk} \geq \underline{x}_{jk} \qquad \forall\, j \in \mathcal{J}, k \in \underline{\mathcal{K}}_j$$

$$x_{jk} \leq \overline{x}_{jk} \qquad \forall\, j \in \mathcal{J}, k \in \overline{\mathcal{K}}_j$$

$$d_0 + \sum_{j \in \mathcal{J}} d_j = 1$$

$$d > 0, \quad d_0 > 0$$

Convex Transformation

Consider the relaxation:

$$\max_{d,\, d_0,\, x} \quad \sum_{j\in\mathcal{J}} \left(\sum_{k\in\mathcal{K}_j} \phi_k x_{jk} - \psi_j \right) d_j \qquad (\mathrm{SP}_3^{\mathrm{MNL}})$$

$$\begin{aligned}
\text{s.t.} \quad & \ln\left(\frac{d_j}{d_0}\right) \leq -\sum_{k\in\mathcal{K}_j} x_{jk} && \forall\, j \in \mathcal{J} \\
& \ln\left(\frac{d_0}{d_j}\right) \leq \sum_{k\in\mathcal{K}_j} \overline{x}_{jk} && \forall\, j \in \overline{\mathcal{J}} \\
& x_{jk} \geq \underline{x}_{jk} && \forall\, j \in \mathcal{J}, k \in \underline{\mathcal{K}}_j \\
& x_{jk} \leq \overline{x}_{jk} && \forall\, j \in \mathcal{J}, k \in \overline{\mathcal{K}}_j \\
& d_0 + \sum_{j\in\mathcal{J}} d_j = 1 \\
& d > 0, \quad d_0 > 0
\end{aligned}$$

Convex Transformation

Observation 1: Second inequality should always hold

$$
\begin{aligned}
\max_{d,\, d_0,\, x} \quad & \sum_{j\in\mathcal{J}} \left(\sum_{k\in\mathcal{K}_j} \phi_k x_{jk} - \psi_j \right) d_j && (\mathrm{SP}_3^{\mathrm{MNL}}) \\
\text{s.t.} \quad & \ln\left(\frac{d_j}{d_0}\right) \leq -\sum_{k\in\mathcal{K}_j} x_{jk} && \forall\, j \in \mathcal{J} \\
& \ln\left(\frac{d_0}{d_j}\right) \leq \sum_{k\in\mathcal{K}_j} \overline{x}_{jk} && \forall\, j \in \overline{\mathcal{J}} \\
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& x_{jk} \leq \overline{x}_{jk} && \forall\, j \in \mathcal{J}, k \in \overline{\mathcal{K}}_j \\
& d_0 + \sum_{j\in\mathcal{J}} d_j = 1 \\
& d > 0, \quad d_0 > 0
\end{aligned}
$$

Observation 2: First inequality always tight at optimality

$$\max_{d,\, d_0,\, x} \quad \sum_{j \in \mathcal{J}} \left(\sum_{k \in \mathcal{K}_j} \phi_k x_{jk} - \psi_j \right) d_j \qquad (\mathrm{SP}_3^{\mathrm{MNL}})$$

$$\text{s.t.} \quad \ln\left(\frac{d_j}{d_0}\right) \leq -\sum_{k \in \mathcal{K}_j} x_{jk} \qquad \forall\, j \in \mathcal{J}$$

$$\ln\left(\frac{d_0}{d_j}\right) \leq \sum_{k \in \mathcal{K}_j} \overline{x}_{jk} \qquad \forall\, j \in \overline{\mathcal{J}}$$

$$x_{jk} \geq \underline{x}_{jk} \qquad \forall\, j \in \mathcal{J}, k \in \underline{\mathcal{K}}_j$$

$$x_{jk} \leq \overline{x}_{jk} \qquad \forall\, j \in \mathcal{J}, k \in \overline{\mathcal{K}}_j$$

$$d_0 + \sum_{j \in \mathcal{J}} d_j = 1$$

$$d > 0, \quad d_0 > 0$$

Convex Transformation

Now, let $u_{jk} = d_j x_{jk}$:

$$
\begin{aligned}
\max_{d,\, d_0,\, u} \quad & \sum_{j \in \mathcal{J}} \left(\sum_{k \in \mathcal{K}_j} \phi_k u_{jk} - \psi_j d_j \right) && (\mathrm{SP}_4^{\mathrm{MNL}}) \\
\text{s.t.} \quad & d_j \ln\left(\frac{d_j}{d_0}\right) \le -\sum_{k \in \mathcal{K}_j} u_{jk} && \forall\, j \in \mathcal{J} \\
& d_0 \ln\left(\frac{d_0}{d_j}\right) \le \sum_{k \in \mathcal{K}_j} \overline{x}_{jk} d_0 && \forall\, j \in \overline{\mathcal{J}} \\
& u_{jk} \ge \underline{x}_{jk} d_j && \forall\, j \in \mathcal{J}, k \in \underline{\mathcal{K}}_j \\
& u_{jk} \le \overline{x}_{jk} d_j && \forall\, j \in \mathcal{J}, k \in \overline{\mathcal{K}}_j \\
& d_0 + \sum_{j \in \mathcal{J}} d_j = 1 \\
& d > 0, \quad d_0 > 0
\end{aligned}
$$

Convex Transformation

Problem: The feasible set is open

$$
\begin{aligned}
\max_{d,\, d_0,\, u} \quad & \sum_{j \in \mathcal{J}} \left(\sum_{k \in \mathcal{K}_j} \phi_k u_{jk} - \psi_j d_j \right) && (\mathrm{SP}_4^{\mathrm{MNL}}) \\
\text{s.t.} \quad & d_j \ln\left(\frac{d_j}{d_0}\right) \leq -\sum_{k \in \mathcal{K}_j} u_{jk} && \forall\, j \in \mathcal{J} \\
& d_0 \ln\left(\frac{d_0}{d_j}\right) \leq \sum_{k \in \mathcal{K}_j} \overline{x}_{jk} d_0 && \forall\, j \in \overline{\mathcal{J}} \\
& u_{jk} \geq \underline{x}_{jk} d_j && \forall\, j \in \mathcal{J}, k \in \underline{\mathcal{K}}_j \\
& u_{jk} \leq \overline{x}_{jk} d_j && \forall\, j \in \mathcal{J}, k \in \overline{\mathcal{K}}_j \\
& d_0 + \sum_{j \in \mathcal{J}} d_j = 1 \\
& d > 0, \quad d_0 > 0
\end{aligned}
$$

Conic Programming Formulation

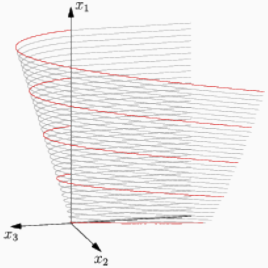
Recall the exponential cone:

$$\begin{aligned}\mathcal{K}_{\text{exp}} &= \text{closure}\big\{(x_1, x_2, x_3) \ : \ x_3 \le x_2 \ln(x_1/x_2), \ x_1 > 0, \ x_2 > 0\big\} \\ &= \big\{(x_1, x_2, x_3) \ : \ x_3 \le x_2 \ln(x_1/x_2), \ x_1 > 0, \ x_2 > 0\big\} \\ &\cup \ \big\{(x_1, 0, x_3) \ : \ x_1 \ge 0, \ x_3 \le 0\big\}\end{aligned}$$

Conic Programming Formulation

Recall the exponential cone:

$$\begin{aligned}\mathcal{K}_{\text{exp}} &= \text{closure}\big\{(x_1, x_2, x_3) \ : \ x_3 \leq x_2 \ln(x_1/x_2), \ x_1 > 0, \ x_2 > 0\big\} \\ &= \big\{(x_1, x_2, x_3) \ : \ x_3 \leq x_2 \ln(x_1/x_2), \ x_1 > 0, \ x_2 > 0\big\} \\ &\cup \ \big\{(x_1, 0, x_3) \ : \ x_1 \geq 0, \ x_3 \leq 0\big\}\end{aligned}$$



Conic Programming Formulation

By taking the closure, our problem becomes:

$$
\begin{aligned}
\max_{d,\, d_0,\, u} \quad & \sum_{j \in \mathcal{J}} \left(\sum_{k \in \mathcal{K}_j} \phi_k u_{jk} - \psi_j d_j \right) && (\mathrm{SP}_5^{\mathrm{MNL}}) \\
\text{s.t.} \quad & \left(d_0, d_j, \sum_{k \in \mathcal{K}_j} u_{jk} \right) \in \mathcal{K}_{\exp} && \forall\, j \in \mathcal{J} \\
& \left(d_j, d_0, -\sum_{k \in \mathcal{K}_j} \overline{x}_{jk} d_0 \right) \in \mathcal{K}_{\exp} && \forall\, j \in \overline{\mathcal{J}} \\
& u_{jk} \geq \underline{x}_{jk} d_j && \forall\, j \in \mathcal{J}, k \in \underline{\mathcal{K}}_j \\
& u_{jk} \leq \overline{x}_{jk} d_j && \forall\, j \in \mathcal{J}, k \in \overline{\mathcal{K}}_j \\
& d_0 + \sum_{j \in \mathcal{J}} d_j = 1
\end{aligned}
$$

Conic Programming Formulation

Result: $d > 0$ **and** $d_0 > 0$ **at optimality.**

$$
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- *Joint Price and Assortment Optimization under Choice Models: A Conic Programming Framework* (with Anton. J. Kleywegt)

Some Related Works

Davis, J., Gallego, G., and Topaloglu, H. (2013).
Assortment Planning under the Multinomial Logit Model with Totally Unimodular Constraint Structures.
Work in Progress.

Dong, L., Kouvelis, P., and Tian, Z. (2009).
Dynamic Pricing and Inventory Control of Substitute Products.
Manufacturing & Service Operations Management, 11(2):317–339.

Gallego, G. and Wang, R. (2014).
Multiproduct Price Optimization and Competition under the Nested Logit Model with Product Differentiated Price Sensitivities.
Operations Research, 62(2):450–461.

Li, H. and Huh, W. T. (2011).
Pricing Multiple Products with the Multinomial Logit and Nested Logit Models: Concavity and Implications.
Manufacturing & Service Operations Management, 13(4):549–563.

Song, J.-S. and Xue, Z. (2007).
Demand Management and Inventory Control for Substitutable Products.
Work in Progress.

Wang, R. (2012).
Joint Optimization of Assortment Selection and Pricing under the Capacitated Multinomial Logit Choice Model with Product-Differentiated Price Sensitivities.
Technical report, Working paper.